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ON BIFURCATION AND STABILITY OF STEADY MOTIONS OF TWO GRAVITATING BODIES†

A. V. KARAPETYAN and I. D. SAKHOKNA

Moscow, Tbilisi

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The plane motion of a system of two mutually gravitating bodies, one a sphere with a spherical mass distribution and the other a homogeneous rod, is considered. All steady motions of the system are found, and the conditions for their stability are obtained in both the secular sense and in the first-order approximation. The possibility of gyroscopic stabilization of steady motions with instability of degree two is noted. The results of the investigation are presented in the form of a bifurcation diagram.

1. CONSIDER the plane motion of two mutually gravitating bodies, one of which is a material point of mass M (or a sphere with a spherical mass distribution), and the other is a homogeneous rod of mass m and length $2a$. The state of the system will be described by the distance r between the mass centres of the bodies, the angle θ between the line joining the centres of mass and some direction fixed in the plane of motion, and the angle φ between the rod and the line joining its midpoint to the point M.

The kinetic and potential energies of the system [1] (f being the gravitational constant)

$$T=\frac{1}{2}\mu M(\dot r^2+r^2\dot\theta^2)+\frac{1}{6}ma^2(\dot\varphi+\dot\theta)^2,\quad \mu=\frac{m}{M+m}$$

$$\Pi=-f\frac{Mm}{2a}\ln\frac{r\cos\varphi+a+\Phi^+}{r\cos\varphi-a+\Phi^-},\ \Phi^\pm=(r^2\pm 2ra\cos\varphi+a^2)^{1/2}$$

do not depend on θ. The system under consideration consequently has the areas integral

$$\partial T/\partial\dot\theta\equiv m(\mu r^2+{}^1\!/_3a^2)\dot\theta+{}^1\!/_3ma^2\dot\varphi=k=\text{const} \tag{1.1}$$

as well as the energy integral $T+\Pi=\text{const}$, and can perform steady motions of the form

$$r=\text{const},\quad \varphi=\text{const},\quad \dot\theta=\omega=\text{const} \tag{1.2}$$

Here the bodies rotate with the same constant angular velocity ω about their common centre of mass, and the rod is stationary with respect to the point M.

Ignoring the cyclic coordinate θ, we introduce the Routhian function

$$R=(T-\Pi-k\dot\theta)\equiv R_2+R_1-W$$

$$R_2=\frac{1}{2}\mu M[\dot r^2+\frac{1}{3}\frac{a^2r^2}{J}\dot\varphi^2],\ R_1=\frac{1}{3}\frac{a}{J}k\dot\varphi$$

$$W=\Pi+\frac{1}{2}\frac{k^2}{J},\quad J=\mu Mr^2+\frac{1}{3}ma^2$$

The steady motion of the original system corresponds to the equilibrium position of the reduced system described by the Routhian function R. Here the constants r and φ in (1.2) are defined by the system

$$\partial W/\partial\varphi=0,\quad \partial W/\partial r=0 \tag{1.3}$$

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The reduced potential W' is π-periodic in φ, and hence we shall study system (1.3) for $0 \leq \varphi \leq \pi$. The first equation of system (1.3) is identically satisfied (with respect to r) when

$$1)\ \varphi = 0,\quad 2)\ \varphi = \pi/2\quad (\mathrm{mod}\,\pi) \tag{1.4}$$

Here the second equation of system (1.3) acquires the form

$$\frac{k^2}{f(M+m)} = F_i(r)\quad (i = 1, 2) \tag{1.5}$$

$$F_1 = \frac{J^2}{(r^2 - a^2)r}\quad (\varphi = 0),\quad F_2 = \frac{J^2}{r^2\sqrt{r^2 + a^2}}\quad \left(\varphi = \frac{\pi}{2}\right)$$

2. Consider steady motion of the form

$$\varphi = 0,\quad r = r_1(k^2) \tag{2.1}$$

where $r_1(k^2)$ is the solution of Eq. (1.5) for $i = 1$. (Here, obviously, it is assumed that $r > a$.)

The function $F_1(r)$ has a lower bound, is positive, tends to infinity both as $r \to +\infty$ and as $r \to a+0$, and reaches a minimum at a unique point $r_{10} > a$ at which its derivative vanishes, where

$$r_{10} = a\left(\frac{4 - 3\mu + \sqrt{(4-3\mu)^2 - 4/3(1-\mu)}}{2(1-\mu)}\right)^{1/2} \tag{2.2}$$

$$(F_1' \gtrless 0\quad \text{for}\quad r \gtrless r_{10}).$$

Thus, if the rod lies on the line connecting its centre to the point M, then for sufficiently small values of the areas integral constant $[k^2 < k_{10}^2 = f(M+m)F_1(r_{10})]$ there are no steady motions (Eq. (1.5) having no solutions for $i = 1$); for $k^2 = k_{10}^2$ a unique solution $r = r_{10}$ exists, while for $k^2 > k_{10}^2$ there are two families of steady motions $r = r_1^+(k^2)$ and $r = r_1^-(k^2)$, with $r_1^+(k^2) > r_{10} > r_1^-(k^2)$.

Computing the coefficients of the second variation matrix of the modified potential at the steady motions (2.1), we have

$$\frac{\partial^2 W}{\partial \varphi^2} = fMm\frac{a^2 r}{(r^2 - a^2)^2} > 0,\quad \forall r,\quad \frac{\partial^2 W}{\partial \varphi \partial r} = 0,$$

$$\frac{\partial^2 W}{\partial r^2} = \mu M\frac{k^2}{(r^2 - a^2)}\frac{F_1'}{F_1^2} \gtrless 0\quad \text{for}\quad r \gtrless r_{10}$$

Thus the family of steady motions $\varphi = 0$, $r = r_1^+(k^2)$ is always stable, while the family of steady motions $\varphi = 0$, $r = r_1^-(k^2)$ is always unstable (the degree of instability being unity).

We also note that the angular velocity of the point M and the rod rotating about their common centre of mass for steady motions (2.1) is given by the relation

$$\omega_1^2 = \left\{\frac{f(M+m)}{r(r^2 - a^2)}\right\}_{r = r_1(k^2)} \tag{2.3}$$

3. We consider steady motions of the form

$$\varphi = \pi/2,\quad r = r_2(k^2) \tag{3.1}$$

where $r_2(k^2)$ is the solution of Eq. (1.5) for $i = 2$ (and here, obviously, it is assumed that $r > 0$).

The function $F_2(r)$ has a lower bound, is positive, tends to infinity both as $r \to +\infty$ and as $r \to 0+$, and reaches a minimum at the unique point $r_{20} > 0$ at which its derivative vanishes, where

$$r_{20} = a\left(\frac{2\mu - 1 + \sqrt{(2\mu-1)^2 + 8/3(1-\mu)}}{2(1-\mu)}\right)^{1/2} \tag{3.2}$$

(here $F_1' \gtrless 0$ for $r \gtrless r_{20}$).

Thus if the rod is perpendicular to the line joining its centre to M, then for

$k^2 < k_{20}^2 = f(m+M)F_2(r_{20})$ there are no steady motions; for $k^2 = k_{20}^2$ a unique solutions exists, while for $k^2 > k_{20}^2$ there are two families of steady motions $r = r_2^+(k^2)$ and $r = r_2^-(k^2)$, with $r = r_2^+(k^2) > r_{20} > r > r_2^-(k^2)$.

Calculating the coefficients of the second variation matrix of the modified potential at the steady motions (3.1), we have

$$\frac{\partial^2 W}{\partial \varphi^2} = -fMm \frac{a^2}{(r^2+a^2)^{3/2}} < 0, \quad \forall r, \quad \frac{\partial^2 W}{\partial \varphi \partial r} = 0$$

$$\frac{\partial^2 W}{\partial r^2} = \mu M \frac{k^2}{r(r^2+a^2)^{1/2}} \frac{F_2'}{F_2^2} \gtrless 0 \quad \text{for} \quad r \gtrless r_{20}$$

Consequently, all steady motions of the form (3.1) are unstable in the secular sense, and the degree of instability of the family $\varphi = \pi/2$, $r = r_2^+(k^2)$ is unity (i.e. this family is Lyapunov-unstable), while the degree of instability of the $\varphi = \pi/2$, $r = r_2^-(k^2)$ family is equal to 2 (i.e. gyroscopic stabilization is possible, see below).

We also note that the angular velocity of the point M and rod rotating about their common centre of mass for steady motions (3.1) is given by the relation

$$\omega_2^2 = \left\{ \frac{f(M+m)}{r^2(r^2+a^2)^{1/2}} \right\}_{r=r_2(k^2)} \tag{3.3}$$

4. We will investigate the possibility of gyroscopic stabilization of the steady motions

$$\varphi = \pi/2, \quad r = r_2^-(k^2) \tag{4.1}$$

whose degree of instability is two.

Linearizing in a neighbourhood of solution (4.1), the equations of perturbed motion of the reduced system can be written in the form

$$Ax'' + Gy' - Cx = 0, \quad By'' - Gx' - Dy = 0 \tag{4.2}$$

$$A = 1, \quad B = \left\{ \frac{r^2}{3(1-\mu)r^2 + a^2} \right\}, \quad G = \left\{ \frac{2a\sqrt{f(M+m)}}{[3(1-\mu)r^2 + a^2](r^2+a^2)^{1/4}} \right\}$$

$$C = f(M+m)\left\{ \frac{2a^4 - 3(1-2\mu)a^2r^2 - 3(1-\mu)r^4}{[3(1-\mu)r^2+a^2]r^2(r^2+a^2)^{3/2}} \right\}, \quad D = \frac{f(M+m)}{\{(r^2+a^2)^{3/2}\}}$$

$$(x = r - r_2^-(k^2), \ y = a(\varphi - \pi/2))$$

The braces indicate that the expressions inside are calculated for $r = r_2^-(k^2)$.

The sufficient conditions for gyroscopic stabilization of the null solution of system (4.1) have the form (see [3])

$$G^2 > AD + BC + 2\sqrt{ABCD} \tag{4.3}$$

from which it follows that gyroscopic stabilization is impossible when $r_2^-(k^2) \to 0$ and clearly occurs when $r_2^-(k^2) \to r_{20}$, $\mu \to 1$. Indeed, relation (4.3) is not satisfied when $r = 0$ and is always satisfied when $r = r_{20}$, $\mu = 1-0$.

In conclusion we note that the system considered cannot perform steady motions other than (1.4), (1.5) because the first equation of system (1.3)

$$\frac{\mu(1-\mu)}{2a} r\sin\varphi \left\{ \frac{\Phi^+ + a}{\Phi^+[\Phi^+ + (r\cos\varphi + a)]} - \frac{\Phi^- - a}{\Phi^-[\Phi^- + (r\cos\varphi - a)]} \right\} = 0 \tag{4.4}$$

is only satisfied by the values $\varphi = 0$, $\pi/2 (\mathrm{mod}\ \pi)$. Indeed, for $\varphi \neq 0$, $\pi/2 (\mathrm{mod}\ \pi)$ Eq. (4.4) is equivalent to the equation $\Phi^+ - \Phi^- + 2a = 0$, which has no solutions.

Furthermore, we note that the critical values r_{10} and r_{20} of the functions F_1 and F_2 satisfy the inequality $r_{10} > r_{20}$, $\mu \in (0, 1)$, while the functions themselves satisfy the inequality $F_1 > F_2$, $r > a$.

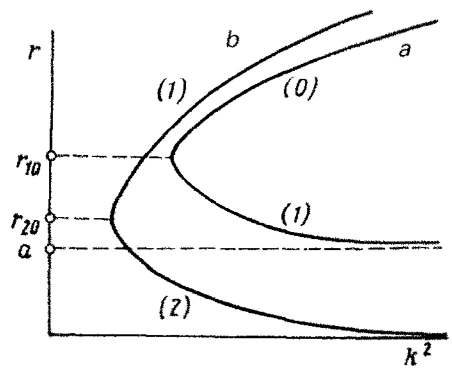

FIG. 1.

Taking the last remarks into account, the results of the investigation can be represented in the form of a birfurcation diagram in the (r, k^2) plane (Fig. 1). Curves a and b correspond to solutions (2.1) and (3.1), and the numbers (0), (1) and (2) denote the degree of instability of the corresponding branches.

REFERENCES

1. SULIKASHVILI R. S., On straightline libration points in the restricted plane problem of three bodies. *Applied Mathematics*, Vol. 1, pp. 358–362. Intern. Acad. Publ., Beijing, 1989.
2. BELETSKII V. V. and PONOMAREVA O. N., Parametric analysis of the stability of relative equilibrium in a gravitational field. *Kosm. Issled.* **28**, 664–675, 1990.
3. CHETAYEV N. G., *Stability of Motion. Papers in Analytical Mechanics*. Izd. Akad. Nauk SSSR, Moscow, 1962.

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